

# Borchers' Commutation Relations and Modular Symmetries

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## Abstract

Recently Borchers has shown that in a theory of local observables, certain unitary and antiunitary operators, which are obtained from an elementary construction suggested by Bisognano and Wichmann, commute with the translation operators like Lorentz boosts and  $P_1CT$ -operators, respectively. We conclude from this that as soon as the operators considered implement *any* symmetry, this symmetry can be fixed up to at most some translation. As a symmetry, we admit any unitary or antiunitary operator under whose adjoint action any algebra of local observables is mapped onto an algebra which can be localized somewhere in Minkowski space.

## 0 Introduction

It is a classical result due to Bisognano and Wichmann that in the traditional (Wightman) theory of quantized fields, Lorentz- and PCT-symmetries are implemented by unitary or antiunitary operators which arise from the Modular Theory due to Tomita and Takesaki [4, 5, 24]. Because of their close relation to the algebraic structure of the theory, these so-called *modular symmetries* are of a particular interest for the algebraic approach to relativistic Quantum Field Theory founded by Haag and Kastler [24]. However, a result like the Bisognano-Wichmann Theorem does not exist for this framework, although the modular objects considered by Bisognano and Wichmann do exist. There are even counterexamples [32] (admittedly, there are, by now, no examples which are Poincaré covariant and satisfy the spectrum condition).

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On the other hand, there are applications and partial results for the algebraic framework. The spin-statistics connection for parabosonic and parafermionic massive particle sectors in the sense of Buchholz and Fredenhagen [17] has recently been derived by different authors from different modular symmetry assumptions [27, 22] (see also [26, 23]); the arguments also work in lower dimensions since any use of the spinor calculus is avoided. In [22], modular  $P_1CT$ -symmetry is, in addition, derived from modular  $\widetilde{\mathcal{P}}_+^\dagger$ -symmetry. For conformal theories of local observables, the Bisognano-Wichmann modular symmetries have been established by different groups in different ways [9, 20, 19]. Conversely, chiral theories in 1+1 dimensions can even be *constructed* by means of their modular objects [31].

These results partly depend on a theorem recently found by Borchers (Theorem II.9 in [7]). The main implication of this theorem is that in (the vacuum sector of) a theory satisfying translation covariance and the spectrum condition, the Bisognano-Wichmann modular objects commute with the translation operators in the same way Lorentz boosts and  $P_1CT$ -operators would commute with these operators. In 1+1 dimensions, Borchers has concluded that each local net of observables satisfying translation covariance and the spectrum condition may be enlarged to a local net of observables satisfying Poincaré covariance and the Bisognano-Wichmann modular symmetry principles. The situation in higher dimensions, however, remained an open problem.

In this paper, we derive from Borchers' commutation relations a uniqueness result which also holds in higher dimensions: Borchers' commutation relations imply that the symmetries which *can* be implemented by the Bisognano-Wichmann modular objects coincide up to at most a translation with the symmetries obtained in the Wightman setting. As a symmetry, we admit any unitary or antiunitary operator under whose adjoint action every algebra of local observables is mapped onto some algebra which may be localized in some open region in Minkowski space. Our results are complementary to those due to Keyl [28] and Araki [2]: in both papers, the notion of a symmetry is more restrictive than ours<sup>1</sup>, whereas both authors avoid the use of the spectrum condition. Araki assumes, in addition, that algebras of local observables which are localized in *timelike* separated regions are not contained

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<sup>1</sup> Our notion of a symmetry (which is made precise below) is a very general one; a symmetry according to our definition need not be a physical one. In fact, we show that any Bisognano-Wichmann modular object which is a symmetry according to our definition is a *physical* symmetry.

in each other's commutants; this property is violated by the massless free field in any even space-time dimension [12, 13, 14, 15].

## 1 Preliminaries and Results

For some integer  $s \geq 1$ , denote by  $\mathbf{R}^{1+s}$  the  $(1+s)$ -dimensional Minkowski space, and let  $V_+$  be the forward light cone. The set  $\mathcal{K}$  of all *double cones*, i.e., the set of all open sets  $\mathcal{O}$  of the form

$$\mathcal{O} := (a + V_+) \cap (b - V_+), \quad a, b \in \mathbf{R}^{1+s},$$

is a convenient topological base of  $\mathbf{R}^{1+s}$ . Each nonempty double cone is fixed by two points, its upper and its lower apex, and the set  $\mathcal{K}$  is invariant under the action of the Poincaré group. For any region  $M$  in Minkowski-space, we denote by  $\mathcal{K}(M)$  the set of all double cones contained in  $M$ .

In the sequel, we denote by  $(\mathcal{H}_0, \mathcal{A})$  a *concrete local net of observables*:  $\mathcal{H}_0$  is a Hilbert space, and the net  $\mathcal{A}$  associates with every double cone  $\mathcal{O} \in \mathcal{K}$  a (concrete)  $C^*$ -algebra  $\mathcal{A}(\mathcal{O})$  consisting of bounded operators in  $\mathcal{H}_0$  and containing the identity operator; this mapping is assumed to be *isotonous*, i.e., if  $\mathcal{O}_1 \subset \mathcal{O}_2$ ,  $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ , then  $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ , and *local*, i.e., if  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are spacelike separated double cones and if  $A \in \mathcal{A}(\mathcal{O}_1)$ ,  $B \in \mathcal{A}(\mathcal{O}_2)$ , then  $AB = BA$ . Since  $\mathcal{K}$  is a topological base of  $\mathbf{R}^{1+s}$ , we may for any open set  $M \subset \mathbf{R}^{1+s}$  consistently define  $\mathcal{A}(M)$  to be the  $C^*$ -algebra generated by the  $C^*$ -algebras  $\mathcal{A}(\mathcal{O})$ ,  $\mathcal{O} \in \mathcal{K}(M)$ . We call  $\tilde{\mathcal{A}} := \mathcal{A}(\mathbf{R}^{1+s})$  the  $C^*$ -algebra of *quasilocal observables*. Note that every state of the normed, involutive algebra  $\mathcal{A}_{\text{loc}} := \bigcup_{\mathcal{O} \in \mathcal{K}} \mathcal{A}(\mathcal{O})$  of *local observables* has a unique continuous extension to a state of  $\tilde{\mathcal{A}}$ .

For any subset  $M$  of  $\mathbf{R}^{1+s}$ , we denote by  $M'$  the *spacelike complement* of  $M$ , i.e., the set of all points in  $\mathbf{R}^{1+s}$  which are spacelike with respect to all points of  $M$ , and for every algebra  $\mathcal{M}$  of bounded operators in some Hilbert space  $\mathcal{H}$ , we denote by  $\mathcal{M}'$  the algebra of all bounded operators which commute with all elements of  $\mathcal{M}$ . Using this notation, the above locality assumption reads  $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}')' \quad \forall \mathcal{O} \in \mathcal{K}$ .

The net  $\mathcal{A}$  is assumed to be covariant under a strongly continuous, unitary representation  $U$  of the translation group, and  $U$  is assumed to satisfy the spectrum condition, i.e., the support of the joint spectral measure  $E$  of the generators of  $U$  is contained in  $\overline{V}_+$ .  $\mathcal{A}$  is assumed to satisfy *weak*

*additivity:*

$$\left( \bigcup_{a \in \mathbf{R}^{1+s}} \mathcal{A}(\mathcal{O} + a) \right)'' = \tilde{\mathcal{A}}'' \quad \forall \mathcal{O} \in \mathcal{K},$$

a useful property which is satisfied by all nets arising from Wightman fields and for which we do not know any interesting counterexample.

We assume the existence and uniqueness up to a phase of a unit vector  $\Omega$  in  $\mathcal{H}_0$  which is invariant under  $U$  and *cyclic* with respect to the concrete algebra  $(\mathcal{H}_0, \tilde{\mathcal{A}})$ , i.e.  $\tilde{\mathcal{A}}\Omega = \mathcal{H}_0$ ;  $\Omega$  will be called the *vacuum vector*. We assume the identical representation  $(\mathcal{H}_0, \text{id}_{\tilde{\mathcal{A}}})$  of  $\tilde{\mathcal{A}}$  to be irreducible, i.e.,

$$\tilde{\mathcal{A}}'' = \mathcal{B}(\mathcal{H}_0),$$

which means that the vacuum state  $A \in \mathcal{A} \mapsto \langle \Omega, A\Omega \rangle$  is pure.

Denote by  $W$  the wedge

$$W := \{x \in \mathbf{R}^{1+s} : x_1 \geq |x_0|\}.$$

It follows from the Reeh-Schlieder Theorem [6] that  $\Omega$  is cyclic with respect to  $(\mathcal{H}_0, \mathcal{A}(W)''')$ , and using a standard argument (see, e.g., Prop. 2.5.3 in [8]), one obtains from locality that  $\Omega$  is also *separating* with respect to  $(\mathcal{H}_0, \mathcal{A}(W)''')$ , i.e., if  $A \in \mathcal{A}(W)'''$  and  $A\Omega = 0$ , then  $A = 0$ . Hence,  $(\mathcal{H}_0, \mathcal{A}(W)''', \Omega)$  is a standard von Neumann algebra. We shall denote by  $J$  and  $\Delta$  the modular conjugation and modular operator, respectively, of this standard von Neumann algebra according to the Tomita-Takesaki analysis [30, 24, 8].

The most general result which has implications for these modular objects is the following theorem due to Borchers:

### 1.1 Theorem (Borchers' commutation relations)

Let  $(\mathcal{H}, \mathcal{M}, \Psi)$  be a standard von Neumann algebra with modular conjugation  $J_{\mathcal{M}}$  and modular operator  $\Delta_{\mathcal{M}}$ , and let  $(T(r))_{r \in \mathbf{R}}$  be a strongly continuous one-parameter group of unitaries which has a positive generator and which for each  $r \geq 0$  satisfies the conditions

- (a)  $T(r)\Psi = Psi;$
- (b)  $T(r)\mathcal{M}T(-r) \subset \mathcal{M}.$

Then for each  $r \in \mathbf{R}$ , the following commutation relations hold:

- (i)  $J_{\mathcal{M}}T(r)J_{\mathcal{M}} = T(-r);$
- (ii)  $\Delta_{\mathcal{M}}^{it}T(r)\Delta_{\mathcal{M}}^{-it} = T(e^{2\pi t}r) \quad \forall t \in \mathbf{R}.$

Together with a standard argument of Tomita-Takesaki Theory, this implies, in our setting, for each  $a \in \mathbf{R}^{1+s}$ :

$$\begin{aligned} (i) \quad JU(a)J &= U(ja), \\ (ii) \quad \Delta^{it}U(a)\Delta^{-it} &= U(V(2\pi t)a) \quad \forall t \in \mathbf{R}. \end{aligned}$$

where

$$j(a_0, a_1, a_2, \dots, a_s) := (-a_0, -a_1, a_2, \dots, a_s), \quad a \in \mathbf{R}^{1+s},$$

and where  $V(2\pi t)$  denotes the Lorentz boost by  $2\pi t$  in the 0-1-plane. So  $J$  and  $\Delta^{it}$ ,  $t \in \mathbf{R}$ , commute with the translations like a P<sub>1</sub>CT-operator and the group of Lorentz boosts in the 0-1-direction, respectively. In 1+1 dimensions, Borchers has derived from this that the net of observables may be enlarged to a local net which generates the same wedge algebras (and, hence, the same corresponding modular objects) as the original one and which has the property that  $J$  is a P<sub>1</sub>CT-operator (*modular P<sub>1</sub>CT-symmetry*), whereas  $\Delta^{it}$  implements the Lorentz boost by  $2\pi t$  for each  $t \in \mathbf{R}$  (*modular Lorentz-symmetry*). This was a first extension of the classical result by Bisognano and Wichmann [4, 5], who established modular P<sub>1</sub>CT- and Lorentz symmetry for Wightman fields in arbitrary spacetime dimensions.

In this paper, we show in particular that  $J$  or  $\Delta^{it}$ ,  $t \in \mathbf{R}$ , can be shown to be, respectively, a P<sub>1</sub>CT-operator or, up to a translation, a 0-1-Lorentz boost as soon as  $J$  or  $\Delta^{it}$  is *any* symmetry. For simplicity, let us first define a *symmetry* to be a unitary or an antiunitary operator  $K$  in  $\mathcal{H}_0$  such that for each  $\mathcal{O} \in \mathcal{K}$ , there is an open set  $M_{\mathcal{O}} \subset \mathbf{R}^{1+s}$  with

$$K\mathcal{A}(\mathcal{O})K^* = \mathcal{A}(M_{\mathcal{O}}).$$

We are ready now to state our main result:

## 1.2 Theorem

Let  $\mathcal{A}$  be as above, let  $K$  be a symmetry, and let  $\kappa : \mathbf{R}^{1+s} \rightarrow \mathbf{R}^{1+s}$  be a linear automorphism such that

$$KU(a)K^* = U(\kappa a) \quad \forall a \in \mathbf{R}^{1+s}.$$

Then there is a unique  $\xi \in \mathbf{R}^{1+s}$  such that

$$K\mathcal{A}(\mathcal{O})K^* = \mathcal{A}(\kappa\mathcal{O} + \xi) \quad \forall \mathcal{O} \in \mathcal{K}.$$

From Theorems 1.1 and 1.2, we shall deduce in particular:

### 1.3 Proposition

(i) If  $J$  is a symmetry, then there is a  $\iota \in \mathbf{R}^{1+s}$  such that

$$J\mathcal{A}(\mathcal{O})J = \mathcal{A}(j\mathcal{O} + \iota) \quad \forall \mathcal{O} \in \mathcal{K}.$$

In more than 1+1 dimensions,  $\iota = 0$ . The same holds in 1+1 dimensions as soon as wedge duality or  $\mathcal{P}_+^\uparrow$ -covariance is given.

(ii) If for some  $t \in \mathbf{R}$ ,  $\Delta^{it}$  is a symmetry, then there is a  $b_t \in \mathbf{R}^{1+s}$  with  $W + b_t = W$  and

$$\Delta^{it}\mathcal{A}(\mathcal{O})\Delta^{-it} = \mathcal{A}(V(2\pi t)\mathcal{O} + b_t).$$

(iii) If for all  $t \in \mathbf{R}$ ,  $\Delta^{it}$  is a symmetry, then there is a  $b \in \mathbf{R}^{1+s}$  with  $W + b = W$  and

$$\Delta^{it}\mathcal{A}(\mathcal{O})\Delta^{-it} = \mathcal{A}(V(2\pi t)\mathcal{O} + bt).$$

(iv) If in a theory in  $1+s \geq 1+3$  dimensions,  $\Delta^{it}$  is a symmetry for some  $t \in \mathbf{R}$  and if some nontrivial rotation  $r$  with  $rW = W$  is implemented by a unitary  $R$  with  $R\Omega = \Omega$ , then the translation  $b_t$  defined in (ii) is trivial.

(v) If in a theory in  $1+s \geq 1+3$  dimensions,  $\Delta^{it}$  is a symmetry for all  $t \in \mathbf{R}$  and in all Lorentz frames, then the vector  $b$  defined in (iii) is trivial.

If in (i), the translation  $\iota$  is trivial, it follows from the results in [21] that  $J$  is indeed not only a  $\underline{P}_1T$ -symmetry, but even a  $\underline{P}_1CT$ -operator.

Conversely,  $\mathcal{P}_+^\uparrow$ -covariance is a sufficient condition for the existence of the rotation  $R$ . Since it has been shown in [10] that the assumption made in (v) implies  $\mathcal{P}_+^\uparrow$ -covariance, (v) is an immediate consequence of (iv).

Proposition 1.3 is not the only application of Theorem 1.2:

### 1.4 Proposition

Assume  $(\mathcal{H}_0, \mathcal{A})$  to be Poincaré covariant, and assume that the vacuum vector  $\Omega$  is not only cyclic, but also separating with respect to the algebra  $(\mathcal{H}_0, \mathcal{A}(V_+)^{\prime\prime})$ .

(i) If the modular conjugation  $J_+$  of  $(\mathcal{H}_0, \mathcal{A}(V_+)'', \Omega)$  is a symmetry, we have

$$J_+ \mathcal{A}(\mathcal{O}) J_+ = \mathcal{A}(-\mathcal{O}) \quad \forall \mathcal{O} \in \mathcal{K}.$$

(ii) If for some  $t \in \mathbf{R}$ , the modular unitary  $\Delta_+^{it}$  of  $(\mathcal{H}_0, \mathcal{A}(V_+)'', \Omega)$  is a symmetry, we have

$$\Delta_+^{it} \mathcal{A}(\mathcal{O}) \Delta_+^{-it} = \mathcal{A}(e^{-2\pi t} \mathcal{O}).$$

Using the scattering theory for massless fermions and bosons in 1+3 or 1+1 dimensions due to Buchholz [12, 13, 14], it has been shown by Buchholz and Fredenhagen [13, 15, 16] that each of the modular symmetries considered in this proposition implies a massless theory to be free (i.e., its S-matrix is trivial).

For the sake of simplicity, we have stated Theorem 1.2 and the Propositions 1.3 and 1.4 in a form which is not yet the most general one: the notion of a symmetry may be generalized considerably. We shall prove our results in the more general form. As a symmetry, we admit any unitary or antiunitary operator  $K$  in  $\mathcal{H}_0$  whose adjoint action preserves at least fragments of the net structure and therefore carries localizable algebras to localizable algebras in the following sense:

## 1.5 Definition

Let  $\beta$  be some set of subalgebras of  $\mathcal{B}(\mathcal{H}_0)$  which contains all local algebras  $\mathcal{A}(\mathcal{O})$ ,  $\mathcal{O} \in \mathcal{K}$ , as elements, and assume that with respect to the partial ordering on  $\beta$  given by set-theoretic inclusion, all  $\mathcal{B} \in \beta$  satisfy  $\mathcal{B} = \sup_{\mathcal{O} \in \mathcal{K}_\mathcal{B}} \mathcal{A}(\mathcal{O})$ , where  $\mathcal{K}_\mathcal{B} := \{\mathcal{O} \in \mathcal{K} : \mathcal{A}(\mathcal{O}) \subset \mathcal{B}\}$ .

For any  $\mathcal{B} \in \beta$ , its *localization region* is given by

$$L(\mathcal{B}) := \bigcup_{\mathcal{O} \in \mathcal{K}_\mathcal{B}} \mathcal{O}.$$

A *symmetry* of  $(\mathcal{H}_0, \mathcal{A})$  with respect to  $\beta$  – or simply: a  $\beta$ -*symmetry* – is a unitary or an antiunitary operator  $K$  in  $\mathcal{H}_0$  with the property that for any  $\mathcal{O} \in \mathcal{K}$ , the algebras  $K\mathcal{A}(\mathcal{O})K^*$  and  $K^*\mathcal{A}(\mathcal{O})K$  are elements of  $\beta$ .

Theorem 1.2 and Propositions 1.3 and 1.4 still hold if the term “symmetry” is replaced by “ $\beta$ -symmetry”. The above formulations are not the only interesting applications: if the local algebras are von Neumann algebras, a possible choice of  $\beta$  is the set of von Neumann algebras generated by local algebras. Furthermore, we need not assume for the elements  $\mathcal{B} \in \beta$  that  $\mathcal{K}_{\mathcal{B}} = \mathcal{K}(L(\mathcal{B}))$ , i.e., not all local algebras localized in the localization region of  $\mathcal{B}$  need to be contained in  $\mathcal{B}$ ;  $L(\mathcal{B}_1) \subset L(\mathcal{B}_2)$  need not imply  $\mathcal{B}_1 \subset \mathcal{B}_2$ .

Before we turn to the proofs of our statements, let us recall a result which relies on arguments due to Jost, Lehmann and Dyson [25, 18] and, in the formulation given here, on results due to Araki [1]. See also [26] for a self-contained exposition of the proof, and see [11] for a slight generalization.

## 1.6 Theorem (Jost, Lehmann, Dyson, Araki)

Let  $K_1$  and  $K_2$  be compact subsets of  $\mathbf{R}^{1+s}$ , and let  $\psi_1$  and  $\psi_2$  be vectors contained in  $E(K_1)\mathcal{H}_0$  and  $E(K_2)\mathcal{H}_0$ , respectively. Let  $A$  and  $B$  be local observables, and assume that the commutator function

$$f_{\psi_1, \psi_2, A, B} : \mathbf{R}^{1+s} \rightarrow \mathbf{C}; \quad x \mapsto \langle \psi_1, [A, U(x)BU(-x)]\psi_2 \rangle$$

vanishes in a Jost-Lehmann-Dyson region, i.e., in an open subset  $R$  of  $\mathbf{R}^{1+s}$  such that each maximal causal curve in  $\mathbf{R}^{1+s}$  intersects the region  $\overline{R} \cup \overline{R}^c$ <sup>2</sup> and such that two continuous functions  $r_+ : \mathbf{R}^s \rightarrow \mathbf{R}$  and  $r_- : \mathbf{R}^s \rightarrow \mathbf{R}$  can be found with the properties:

- (i)  $R = \{x =: (x_0, \vec{x}) \in \mathbf{R}^{1+s} : r_-(\vec{x}) < x_0 < r_+(\vec{x})\}$ ;
- (ii)  $|r_+(\vec{x}) - r_+(\vec{y})| \leq \|\vec{x} - \vec{y}\|_2 \quad \forall \vec{x}, \vec{y} \in \mathbf{R}^s$ ;
- (iii)  $|r_-(\vec{x}) - r_-(\vec{y})| \leq \|\vec{x} - \vec{y}\|_2 \quad \forall \vec{x}, \vec{y} \in \mathbf{R}^s$ .

Then the support of  $f_{\psi_1, \psi_2, A, B}$  is contained not only in the complement of  $R$ , but even in the (in general, smaller) union of all admissible hyperboloids<sup>3</sup>, i.e., the hyperboloids  $H_{a, \sigma} := \{x \in \mathbf{R}^{1+s} : (x - a)^2 = \sigma^2\}$ ,  $a \in \mathbf{R}^{1+s}$ ,  $\sigma \in \mathbf{R}$ , which do not intersect  $R$ .

Another result we shall make use of follows from the Asgeirsson-Lemma, which describes the behaviour of solutions of the wave equation. For a proof, see [1]:

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<sup>2</sup>This assumption is avoided in [11].

<sup>3</sup>in 1+1 dimensions: admissible hyperbolae

### 1.7 Lemma (Asgeirsson, Araki)

*If the commutator function  $f_{\psi_1, \psi_2, A, B}$  defined in Theorem 1.6 and its partial derivatives vanish along a timelike curve segment  $\gamma$ , they also vanish in the entire double cone  $\gamma''$ .*

Note that both these results hold in all dimensions  $1+s \geq 1+1$  and they also apply to the restriction of  $f_{\psi_1, \psi_2, A, B}$  to some timelike plane (a plane which is invariant under translations in some timelike direction).

## 2 Proofs

We first show that in  $1+s \geq 1+2$  dimensions,  $L(\mathcal{A}(\mathcal{O})) = \mathcal{O}$  for every double cone  $\mathcal{O} \in \mathcal{K}$ . This immediately follows from a Theorem due to Landau [29]. The identity to be established is satisfied by any additive theory in  $1+s \geq 1+2$  dimensions, it need not hold in  $(1+1)$ -dimensional theories; the chiral theories are well-known counterexamples. For the proof of Proposition 1.3 (i), we shall need the following, slightly strengthened version of Landau's result:

### 2.1 Theorem

*Let  $\mathcal{O}$  be a double cone whose closure does not intersect the closure of  $W$ . If  $1+s \geq 1+2$  or if  $\mathcal{O}$  is spacelike with respect to  $W$ , we have*

$$\mathcal{A}(\mathcal{O}')' \cap \mathcal{A}(W')' = \mathbf{C} \text{id}.$$

**Proof:** Let  $\mathcal{C}$  be a convex open set such that  $\tilde{\mathcal{O}} := \mathcal{O} - \mathcal{C}$  and  $\tilde{W} := W - \mathcal{C}$  still have disjoint or spacelike separated closures, respectively. Pick  $A \in \mathcal{A}(\mathcal{O}')' \cap \mathcal{A}(W')'$  and  $B \in \mathcal{A}(\mathcal{C})$ , and let  $\psi_1, \psi_2$  and  $f_{\psi_1, \psi_2, A, B}$  be as in Theorem 1.6. It follows from locality that  $f_{\psi_1, \psi_2, A, B}$  vanishes in the region  $\tilde{\mathcal{O}}' \cup \tilde{W}'$ .

This region does not admit any admissible hyperboloid since for any two points  $x$  and  $y$  in  $\mathbf{R}^{1+s}$ , the set  $(x + \overline{V}) \cap (y + \overline{V})$ <sup>4</sup> contains some full mass hyperboloid if and only if  $x = y$  and since

$$\mathbf{R}^{1+s} \setminus (\tilde{\mathcal{O}}' \cup \tilde{W}') = (\tilde{\mathcal{O}} + \overline{V}) \cap (\tilde{W} + \overline{V}).$$

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<sup>4</sup> $\overline{V}$  denotes the closure of the light cone  $V := \{x \in \mathbf{R}^{1+s} : x^2 > 0\}$ .

One verifies that, since  $\tilde{\mathcal{O}}$  and  $\tilde{W}$  are disjoint,  $\tilde{\mathcal{O}}' \cup \tilde{W}'$  is a Jost-Lehmann-Dyson region if and only if  $\tilde{\mathcal{O}}$  is spacelike with respect to  $\tilde{W}$ . In this case, it follows from Theorem 1.6 that  $f_{\psi_1, \psi_2, A, B} \equiv 0$ . Since  $A$  and  $B$  are bounded and the vectors with compact momentum space support are dense in  $\mathcal{H}_0$ , one obtains  $[A, U(x)BU(-x)] = 0 \ \forall x \in \mathbf{R}^{1+s}$ . Since  $B \in \mathcal{A}(\mathcal{C})$  is arbitrary, weak additivity gives

$$A \in \left( \bigcup_{x \in \mathbf{R}^{1+s}} \mathcal{A}(\mathcal{C} + x) \right)' = \tilde{\mathcal{A}}' = \mathcal{B}(\mathcal{H}_0)' = \mathbf{C} \text{ id},$$

as stated.

Assume now that  $\mathcal{O}$  is not necessarily spacelike with respect to  $W$  and that  $1+s \geq 1+2$ . Sufficiently large shifts  $Q+a$  of the plane  $Q := \{x \in \mathbf{R}^{1+s} : x_2 = \dots = x_s = 0\}$  have the property that, with respect to the causal structure the manifold  $Q+a$  inherits from Minkowski space, the region  $(Q+a) \cap (\tilde{\mathcal{O}}' \cup \tilde{W}')$  is a Jost-Lehmann-Dyson region. In particular, the set  $\alpha$  of all  $a \in \mathbf{R}^{1+s}$  such that  $Q+a$  is of this kind, contains an open subset of  $\mathbf{R}^{1+s}$ . One verifies that for any shift  $Q+a$  of  $Q$ , the region  $(Q+a) \cap (\tilde{\mathcal{O}}' \cup \tilde{W}')$  does not admit any admissible hyperbola. Theorem 1.6 now implies for every  $a \in \alpha$  that  $f_{\psi_1, \psi_2, A, B}|_{Q+a} \equiv 0$ . Since  $\alpha$  contains an open region, we conclude that  $f_{\psi_1, \psi_2, A, B}$  vanishes in a neighborhood of  $Q+a$  for some  $a \in \alpha$ . Lemma 1.7 now implies that  $f_{\psi_1, \psi_2, A, B} \equiv 0$  all over  $\mathbf{R}^{1+s}$ , from which  $A \in \mathbf{C} \text{ id}$  is obtained as above.

□

The theorem is stated in Poincaré invariant terms, so  $W$  and  $\mathcal{O}$  may be replaced by their respective images under any Poincaré transform. Note furthermore that the proof still works if  $\mathcal{O}$  is replaced by a causally complete spacelike cone in the sense of [17]. The following lemma is taken from [3].

## 2.2 Lemma

*Assume that  $1+s \geq 1+2$ . For any two double cones  $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ ,  $\mathcal{A}(\mathcal{O}_1)$  is a subalgebra of  $\mathcal{A}(\mathcal{O}_2)$  if and only if  $\mathcal{O}_1 \subset \mathcal{O}_2$ .*

**Proof:** Because of isotony, we only need to show that the condition is necessary. If  $\mathcal{O}_1$  is not contained in  $\mathcal{O}_2$  as a subset, there is a double cone  $\mathcal{O}$  that

is contained in  $\mathcal{O}_1$  and whose closure does not intersect the closure of  $\mathcal{O}_2$ . Theorem 2.1 implies that  $\mathcal{A}(\mathcal{O}_2) \cap \mathcal{A}(\mathcal{O}) = \mathbf{C} \text{id}$ . It follows from additivity that  $\mathcal{A}(\mathcal{O}) \neq \mathbf{C} \text{id}$ , so  $\mathcal{A}(\mathcal{O})$  is not a subset of  $\mathcal{A}(\mathcal{O}_2)$ . Since  $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}_1)$  follows from isotony,  $\mathcal{A}(\mathcal{O}_1)$  cannot be a subset of  $\mathcal{A}(\mathcal{O}_2)$ .

□

This implies that in 1+2 or more dimensions,  $L(\mathcal{A}(\mathcal{O})) = \mathcal{O}$  for every double cone  $\mathcal{O} \in \mathcal{K}$ .

### 2.3 Proof of Theorem 1.2

In the sequel,  $\beta$ ,  $K$  and  $\kappa$  are defined as in Definition 1.5 and Theorem 1.2. We subdivide the proof into a couple of Lemmas.

#### 2.3.1 Lemma

For any algebra  $\mathcal{B} \in \beta$ , we have

$$L(K\mathcal{B}K^*) - L(K\mathcal{B}K^*) = \kappa(L(\mathcal{B}) - L(\mathcal{B})).$$

**Proof:** We have for any  $\mathcal{B} \in \beta$ :

$$\begin{aligned} L(\mathcal{B}) - L(\mathcal{B}) &= \{a \in \mathbf{R}^{1+s} : \exists x \in L(\mathcal{B}) : x + a \in L(\mathcal{B})\} \\ &= \{a \in \mathbf{R}^{1+s} : \exists \mathcal{O} \in \mathcal{K}(L(\mathcal{B})) : \mathcal{O} + a \subset L(\mathcal{B})\} \\ &= \{a \in \mathbf{R}^{1+s} : \exists \mathcal{O} \in \mathcal{K}(L(\mathcal{B})) : U(a)\mathcal{A}(\mathcal{O})U(-a) \subset \mathcal{B}\} \\ &= \{a \in \mathbf{R}^{1+s} : \exists \mathcal{O} \in \mathcal{K}_{\mathcal{B}} : U(a)\mathcal{A}(\mathcal{O})U(-a) \subset \mathcal{B}\}. \end{aligned}$$

Using this and the assumptions of Theorem 1.2, we conclude

$$\begin{aligned} L(K\mathcal{B}K^*) - L(K\mathcal{B}K^*) &= \{a \in \mathbf{R}^{1+s} : \exists \mathcal{O} \in \mathcal{K}_{K\mathcal{B}K^*} : U(a)\mathcal{A}(\mathcal{O})U(-a) \subset K\mathcal{B}K^*\} \\ &= \{a \in \mathbf{R}^{1+s} : \exists \mathcal{O} \in \mathcal{K}_{K\mathcal{B}K^*} : KU(\kappa^{-1}a)K^*\mathcal{A}(\mathcal{O})KU(-\kappa^{-1}a)K^* \subset K\mathcal{B}K^*\} \\ &= \kappa\{a \in \mathbf{R}^{1+s} : \exists \mathcal{O} \in \mathcal{K}_{K\mathcal{B}K^*} : U(a)K^*\mathcal{A}(\mathcal{O})KU(-a) \subset \mathcal{B}\} \\ &\subset \kappa\{a \in \mathbf{R}^{1+s} : \exists \mathcal{O} \in \mathcal{K}_{K\mathcal{B}K^*} : \exists \tilde{\mathcal{O}} \in \mathcal{K}_{K^*\mathcal{A}(\mathcal{O})K} : U(a)\mathcal{A}(\tilde{\mathcal{O}})U(-a) \subset \mathcal{B}\} \\ &\subset \kappa\{a \in \mathbf{R}^{1+s} : \exists \tilde{\mathcal{O}} \in \mathcal{K}_{\mathcal{B}} : U(a)\mathcal{A}(\tilde{\mathcal{O}})U(-a) \subset \mathcal{B}\} \\ &= \kappa(L(\mathcal{B}) - L(\mathcal{B})). \end{aligned}$$

If we replace  $\mathcal{B}$  by  $K^* \mathcal{B} K$  and  $K$  by  $K^*$ , the same reasoning yields the converse inclusion:

$$\begin{aligned}\kappa(L(\mathcal{B}) - L(\mathcal{B})) &= \kappa(L(K^* K \mathcal{B} K^* K) - L(K^* K \mathcal{B} K^* K)) \\ &\subset \kappa(\kappa^{-1}(L(K \mathcal{B} K^*) - L(K \mathcal{B} K^*))) \\ &= L(K \mathcal{B} K^*) - L(K \mathcal{B} K^*).\end{aligned}$$

This proves the Lemma.  $\square$

### 2.3.2 Lemma

For any two double cones  $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$  with  $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$ , we have

$$\overline{L(K\mathcal{A}(\mathcal{O}_1)K^*)} \subset L(K\mathcal{A}(\mathcal{O}_2)K^*).$$

**Proof:**  $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$  if and only if the set

$$\{a \in \mathbf{R}^{1+s} : \mathcal{O}_1 + a \subset \mathcal{O}_2\}$$

is an open neighbourhood of the origin of  $\mathbf{R}^{1+s}$ . Since

$$\begin{aligned}\{a \in \mathbf{R}^{1+s} : \mathcal{O}_1 + a \subset \mathcal{O}_2\} &= \{a \in \mathbf{R}^{1+s} : U(a)\mathcal{A}(\mathcal{O}_1)U(-a) \subset \mathcal{A}(\mathcal{O}_2)\} \\ &= \{a \in \mathbf{R}^{1+s} : K^*U(\kappa^{-1}a)K\mathcal{A}(\mathcal{O}_1)K^*U(-\kappa^{-1}a)K \subset \mathcal{A}(\mathcal{O}_2)\} \\ &= \{a \in \mathbf{R}^{1+s} : U(\kappa^{-1}a)K\mathcal{A}(\mathcal{O}_1)K^*U(-\kappa^{-1}a) \subset K\mathcal{A}(\mathcal{O}_2)K^*\} \\ &= \kappa\{a \in \mathbf{R}^{1+s} : U(a)K\mathcal{A}(\mathcal{O}_1)K^*U(-a) \subset K\mathcal{A}(\mathcal{O}_2)K^*\} \\ &\subset \kappa\{a \in \mathbf{R}^{1+s} : L(K\mathcal{A}(\mathcal{O}_1)K^*) + a \subset L(K\mathcal{A}(\mathcal{O}_2)K^*)\},\end{aligned}$$

and since  $\kappa$  is a homeomorphism of  $\mathbf{R}^{1+s}$  onto itself, it follows that this set is an open neighbourhood of the origin if and only if

$$\{a \in \mathbf{R}^{1+s} : L(K\mathcal{A}(\mathcal{O}_1)K^*) + a \subset L(K\mathcal{A}(\mathcal{O}_2)K^*)\}$$

is an open neighbourhood of the origin. This gives the statement.  $\square$

### 2.3.3 Lemma

Let  $x \in \mathbf{R}^{1+s}$  be arbitrary, and let  $(\mathcal{O}_\nu)_{\nu \in \mathbf{N}}$  be a neighbourhood base of  $x$  consisting of double cones  $\mathcal{O}_\nu \in \mathcal{K}$ .

Then  $(L(K\mathcal{A}(\mathcal{O}_\nu)K^*))_{\nu \in \mathbf{N}}$  is a neighbourhood base of a (naturally, unique) point  $\tilde{\kappa}(x) \in \mathbf{R}^{1+s}$ .

**Proof:** Without loss of generality, we may assume that  $\overline{\mathcal{O}_{\nu+1}} \subset \mathcal{O}_\nu \quad \forall \nu \in \mathbf{N}$ . It follows from  $L(\mathcal{A}(\mathcal{O})) = \mathcal{O} \quad \forall \mathcal{O} \in \mathcal{K}$  and Lemma 2.3.1 that all  $L(K\mathcal{A}(\mathcal{O}_\nu)K^*)$ ,  $\nu \in \mathbf{N}$ , are bounded sets, and it follows from Lemma 2.3.2 that

$$\overline{L(K\mathcal{A}(\mathcal{O}_{\nu+1})K^*)} \subset L(K\mathcal{A}(\mathcal{O}_\nu)K^*).$$

Therefore, the intersection of this family is nonempty, and Lemma 2.3.1 implies that the diameter of  $L(K\mathcal{A}(\mathcal{O}_\nu)K^*)$  tends to zero as  $\nu$  tends to infinity. This implies that the intersection contains precisely one point  $\tilde{\kappa}(x)$ , as stated.

□

### 2.3.4 Corollary

The map  $\tilde{\kappa}$  given by Lemma 2.3.3 is a homeomorphism, and for any algebra  $\mathcal{B} \in \beta$ , we have

$$L(K\mathcal{B}K^*) = \tilde{\kappa}(L(\mathcal{B})).$$

### 2.3.5 Lemma

For each  $x \in \mathbf{R}^{1+s}$ , we have

$$\tilde{\kappa}(x) = \tilde{\kappa}(o) + \kappa x,$$

where  $o$  denotes the origin of  $\mathbf{R}^{1+s}$ .

**Proof:** Let  $(\mathcal{O}_\nu)_{\nu \in \mathbf{N}}$  be a neighborhood base of  $o$ . Then  $(\mathcal{O}_\nu + x)_{\nu \in \mathbf{N}}$  is a neighborhood base of  $x$ , and

$$\bigcap_{\nu \in \mathbf{N}} L(K\mathcal{A}(\mathcal{O}_\nu + x)K^*) = \bigcap_{\nu \in \mathbf{N}} \tilde{\kappa}(\mathcal{O}_\nu + x) = \{\tilde{\kappa}(x)\}.$$

On the other hand, we have

$$\begin{aligned}
\bigcap_{\nu \in \mathbf{N}} L(K\mathcal{A}(\mathcal{O}_\nu + x)K^*) &= \bigcap_{\nu \in \mathbf{N}} L(U(\kappa x)K\mathcal{A}(\mathcal{O}_\nu)K^*U(-\kappa x)) \\
&= \kappa x + \bigcap_{\nu \in \mathbf{N}} \tilde{\kappa}(\mathcal{O}_\nu) \\
&= \kappa x + \{\tilde{\kappa}(o)\}.
\end{aligned}$$

□

This proves Theorem 1.2, where  $k = \tilde{\kappa}(o)$ .

## 2.4 Proof of the Propositions 1.3 and 1.4

To start with (i) of Proposition 1.3, it follows from Theorem 1.2 that there is a homeomorphism  $\tilde{j}$  of  $\mathbf{R}^{1+s}$  into itself such that

$$J\mathcal{A}(\mathcal{O})J = \mathcal{A}(\tilde{j}(\mathcal{O})) \quad \forall \mathcal{O} \in \mathcal{K}.$$

This  $\tilde{j}$  satisfies the relation

$$\tilde{j}(x) = \tilde{j}(o) + jx =: \iota + jx \quad \forall x \in \mathbf{R}^{1+s}.$$

We have to show that  $\iota = o$ . Since  $J$  is an involution, so is  $\tilde{j}$ . This implies

$$x = \tilde{j}(\tilde{j}(x)) = \tilde{j}(\iota + jx) = \iota + j\iota + x \quad \forall x \in \mathbf{R}^{1+s},$$

which gives  $\iota = -j\iota$ , hence  $\iota_2 = \dots = \iota_s = 0$ . It remains to show that  $\iota_0 = \iota_1 = 0$ , which is equivalent to  $\tilde{j}(W) = W'$ . To this end, note that  $W' \subset \tilde{j}(W)$  already follows from locality. If wedge duality is assumed, the converse inclusion, which completes the proof, is trivial, and if, instead,  $\mathcal{P}_+^\uparrow$ -covariance is assumed, the statement immediately follows from the fact that the 0-1-boosts commute with  $J$  and, therefore, leave the algebra  $J\mathcal{A}(W)''J = \mathcal{A}(\tilde{j}(W))''$  invariant.

Now let us just assume that  $1+s \geq 1+2$  and that  $\tilde{j}(W) \not\subset W'$ . Then there is a double cone  $\mathcal{O} \in \mathcal{K}(\tilde{j}(W))$  whose closure is disjoint from  $\overline{W'}$ . Because of

$$\mathcal{K}(\tilde{j}(W)) = \mathcal{K}_{J\mathcal{A}(W)''J} = \mathcal{K}_{\mathcal{A}(W)'},$$

we conclude  $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(W)'$ . Theorem 2.1 now implies that  $\mathcal{A}(\mathcal{O}) = \mathbf{C} \text{id}$ , and by additivity, the whole net has to be trivial. This, however, is in conflict with irreducibility, which implies  $\tilde{j}(W) = W'$ .

(ii) and (iii) are immediate consequences of Theorem 1.2.  
(iv) follows from Theorem 2.1 since for each  $\mathcal{O} \in \mathcal{K}$ , we have, on the one hand:

$$R\Delta^{it}\mathcal{A}(\mathcal{O})\Delta^{-it}R^* = R\mathcal{A}(V(2\pi t\mathcal{O} + b_t)R^* = \mathcal{A}(rV(2\pi t)\mathcal{O} + rb_t),$$

whereas on the other hand,  $R\mathcal{A}(W)''R^* = \mathcal{A}(W)''$  and  $R\Omega = \Omega$  imply  $R\Delta^{it} = \Delta^{it}R$  so that

$$R\Delta^{it}\mathcal{A}(\mathcal{O})\Delta^{-it}R^* = \Delta^{it}\mathcal{A}(r\mathcal{O})\Delta^{-it} = \mathcal{A}(V(2\pi t)r\mathcal{O} + b_t) = \mathcal{A}(rV(2\pi t)\mathcal{O} + b_t);$$

comparing the two expressions and using  $L(\mathcal{A}(\mathcal{O})) = \mathcal{O} \quad \forall \mathcal{O} \in \mathcal{K}$  yields the stated result.

(v) has been derived from (iv) in the remark following Proposition 1.3.  
To prove Proposition 1.4, note that Theorem 1.1 implies for all  $a \in \mathbf{R}^{1+s}$  the commutation relations

$$\begin{aligned} J_+U(a)J_+ &= U(-a); \\ \Delta_+^{it}U(a)\Delta_+^{-it} &= U(e^{2\pi t}a) \quad \forall t \in \mathbf{R}. \end{aligned}$$

If, respectively  $J_+$  or  $\Delta_+^{it}$  is a symmetry, Theorem 1.2 implies that it can differ from the stated symmetry at most by a translation. Since  $V_+$  is Lorentz-invariant, the modular data of  $(\mathcal{H}_0, \mathcal{A}(V_+)'', \Omega)$  commute with all  $U(g)$ ,  $g \in \mathcal{P}_+^\uparrow$ . However, there are no nontrivial translations which commute with all  $g \in \mathcal{P}_+^\uparrow$ ; this proves Proposition 1.4.

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